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Thermal History of the Universe:

Currently, we can probe two epochs in the history of the universe from observations directly. From the CMB experiments, we get a snapshot of the epoch of recombination, when the universe was ~400,000 old. From the measurement of light elements, we get information about the epoch of big-bang nucleosynthesis (BBN) when the universe was 1 sec - 3 min old.

These epochs (and some other ones, not directly probed by experiment at this point) are representatives of these special moments in the history of the universe when certain types of interactions are rendered ineffective. At $t \sim 1$ sec, the weak interactions drop out of equilibrium as a result of which neutrinos decouple from the rest of particles in the universe

At $t \sim 400{,}000$ yr, neutral atoms are formed as a result of which photons decouple and start moving freely. In order to study these epochs (and other, more speculative, ones) we need to know the rate for various interactions among particles in an expanding universe.

Thermodynamics in an Expanding Universe:

First, let us get a feeling of the number of particles that we are dealing with. The density of CMB photons can be calculated from their temperature $T = 2.72^\circ K$, which implies a number density $n_\gamma = 410\ cm^{-3}$ at the present time. The total number of photons in the observable part of the universe, which has a radius ~ 13.8 Glyr, is then found to be $\sim 10^{87}$. As we will see later, the number of baryons is smaller than this by a factor of $O(10^{9})$, which results in a number of $O(10^{78})$.

Since baryons are predominantly in the form of proton, which we will see later, charge neutrality of the universe implies a number of $O(10^{78})$ for the electrons too. The sheer size of this system consisting of a huge number of particles tells us that we deal with a thermodynamic system. For such a large ensemble of particles, occupation number $f(\vec{p},\vec{x})$ is an appropriate quantity. It is related to th number of particles within a phase space volume $d^3p\, d^3x$ about point $(\vec{p},\vec{x})$ according to ;

$$dN = \frac{f(\vec{p},\vec{x},t)}{(2\pi\hbar)^3}\, d^3p\, d^3x$$

In a homogeneous and isotropic universe the probability to find a particle in a volume d^3x does not depend on the position $\vec{x}$, and hence f is a function of $\vec{p}$ only. Moreover, f is a function of $p \equiv |\vec{p}|$ only because of the isotropicity.

In general, time dependence of f makes the situation quite complicated. However, if a system reaches equilibrium, then f will not have an explicit dependence on t. Reaching equilibrium requires the existence of interactions among the particles that drive the system from an arbitrary initial condition toward equilibrium. In the (early) universe the interactions among particles are governed by the three fundamental forces: electromagnetism, weak, and strong. The time scale for these interactions to drive particles toward equilibrium is set by the strength of the fundamental forces. Since the universe expands, it is not guaranteed that particles will reach equilibrium in any case. Intuitively, equilibrium can be achieved if the relevant time scale is shorter than that set by the Hubble expansion rate. The latter is governed by gravity, which is by far the weakest of the forces of nature. This implies that at sufficiently early times, we can expect the time scale for interactions among particles to be (much) shorter than that for the universe expansion. In consequence, particles can reach a thermal equilibrium distribution quickly. Of course, this is a heuristic argument, which will be quantified later on. Nevertheless, it justifies thermal equilibrium condition in the (very) early universe.

Under such condition, the occupation number f will be considerably simplified. It will be a function of the temperature T and p according to:

$$f(p,T) = \frac{1}{\exp\left(\frac{E-\mu}{k_B T}\right) \mp 1} \qquad E = \sqrt{p^2+m^2}\ , \ k_B\text{: Boltzmann Constant}$$

Here μ is the chemical potential, and $-$ $(+)$ sign in the denominator is used for bosons (fermions). The chemical potential is associated with (some kind of) conservation of the number of particles. For simplicity, we neglect μ for the time being, which is justified for particles like photons whose number is not subject to any conservation law (for example, note that $e^+ + e^- \leftrightarrow \gamma\gamma$ processes can create or annihilate photons).

In this case;

$$f(p,T) = \frac{1}{\exp\left(\frac{\sqrt{p^2+m^2}}{k_B T}\right) \mp 1} \qquad \begin{array}{l} -:\ \text{boson} \\ +:\ \text{fermion} \end{array}$$

In the relativistic regime $T \gg m$, the mass can be neglected, leading to:

$$f(p,T) = \frac{1}{\exp\left(\frac{p}{k_B T}\right) \mp 1}$$

The average number density of particles is then given by

$$n(T) = \frac{1}{(2\pi\hbar)^3} \int f(p,T)\, d^3p = \frac{1}{(2\pi\hbar)^3} \int f(p,T)\, p^2 dp\, d\Omega$$

Because of isotropicity, integrating over Ω simply gives a factor 4π. Using natural units (where $\hbar = k_B = 1$), we find:

$$n_{(T)} = \frac{\zeta(3)}{\pi^2} T^3 \quad \text{bosons}$$

$$n_{(T)} = \frac{3}{4} \frac{\zeta(3)}{\pi^2} T^3 \quad \text{fermion} \qquad (I)$$

Here $\zeta(3) = 1.20206\ldots$ is the zeta function of 3, where the Riemann Zeta function is defined as $\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s}$. Similar one can find the average energy density:

$$\rho_{(T)} = \frac{1}{(2\pi)^3} \int f(p,T)\, E\, d^3p$$

In the relativistic regime, this becomes:

$$\rho_{(T)} = \frac{\pi^2}{30} T^4 \quad \text{bosons}$$

$$\rho_{(T)} = \frac{7}{8} \frac{\pi^2}{30} T^4 \quad \text{fermions} \qquad (II)$$

The expressions in Eqs. (I, II) are given for a single degree of freedom that is in thermal equilibrium at temperature T. A degree of freedom is identified by a set of quantum numbers (e.g., mass, charge, spin, etc)

In general, for g_B bosonic degrees of freedom and g_F fermionic degrees of freedom (in the relativistic regime) we have:

$$n(T) = \left(g_B + \frac{3}{4} g_F\right) \frac{\zeta(3)}{\pi^2} T^3$$

$$\rho(T) = \left(g_B + \frac{7}{8} g_F\right) \frac{\pi^2}{30} T^4 \qquad (\mathrm{III})$$

To give enamples, there are two degrees of freedom associated with the photon corresponding to its two polarizations. Four degrees of freedom are associated to the electron, which correspond to two spin states for particles (electrons) and two spin states for antiparticles (positrons). Two degrees of freedom are associated with each flavor of neutrios corresponding to one spin state for particles (neutrinos) and antiparticles (antineutrinos) each.